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THE PROBLEM OF AN ANNULAR CRACK AT THE INTERFACE BETWEEN AN ELASTIC LAYER AND AN ELASTIC HALF-SPACE†

V. M. ALEKSANDROV and D. A. POZHARSKII

Moscow and Rostov-on-Don

e-mail: alexand@ipmnet.ru

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The axisymmetric problem of the annular crack that occurs at the interface of an elastic layer bonded to an elastic half-space of a different material when a normal load, distributed around a circular region, is applied to the surface of such a two-layer body is investigated. To solve the system of integral equations for the sudden changes in the components of the displacement vector in the crack zone, the regular asymptotic method [1] is used, which is effective for a fairly narrow annular crack. After the system of equations has been solved, to determine the true dimensions of the crack it is suggested that Novozhilov's integral criterion of fracture [2] be used.

Problems of annular defects of the crack or inclusion type in a plane where the elastic properties of the materials change [3, 4], and also similar plane problems, have been investigated earlier, for example in [5].

1. FORMULATION OF THE PROBLEM AND ITS REDUCTION TO A SYSTEM OF TWO INTEGRAL EQUATIONS

Consider an elastic body consisting of a layer of thickness h with elastic characteristics G_1 (shear modulus) and υ_1 (Poisson's ratio) bonded to a half-space with characteristics G_2 and υ_2. Under conditions of axial symmetry, we will use cylindrical coordinates r, φ, z. Suppose a normal load of intensity $q(r)$, distributed around a circle of radius c, is applied to the boundary of the layer $z = h$. Outside this circle, the body surface is assumed to be unloaded. We will assume that, under the action of the applied normal load in the plane $z = 0$ in which the elastic properties change, because of inadequate adhesion between the layer (coating) and the half-space (substrate), an annular crack $a \leqslant r \leqslant b$ is formed (note that an annular crack could be formed under the action of an annular load). The normal and radial displacements are interrupted at the crack. Assuming firstly that the values of a and b are known, we will obtain a system of integral equations for the sudden changes in displacements at the crack.

We will write the boundary conditions of the problem in the following form (superscript 1 corresponds to the layer, and superscript 2 to the half-space)

$$
\begin{aligned}
&z = h: \ \sigma_z^{(1)} = -q(r)(r \leqslant c),\ \sigma_z^{(1)} = 0(r > c);\ \tau_{rz}^{(1)} = 0\\
&z = 0: \ u_r^{(1)} - u_r^{(2)} = u(r)(r \in (a,b));\ u_z^{(1)} - u_z^{(2)} = w(r)(r \in (a,b))\\
&u_r^{(1)} = u_r^{(2)}(r \notin (a,b));\ u_z^{(1)} = u_z^{(2)}(r \notin (a,b))\\
&\sigma_z^{(1)} = \sigma_z^{(2)};\ \tau_{rz}^{(1)} = \tau_{rz}^{(2)};\ \sigma_z^{(1)} = \tau_{rz}^{(1)} = 0(r \in (a,b))
\end{aligned}
\tag{1.1}
$$

Furthermore as $z \to -\infty$, the displacements and stresses disappear. To solve boundary-value problem (1.1), we will use four Papkovich–Neuber functions $\Phi_n^m(r,z)$ $(n, m = 1, 2)$: two for the layer (superscript 1) and two for the half-space (superscript 2). The displacements and stresses are expressed in terms of these functions by means of formulae (59.3) in [6].

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Assuming that

$$
\begin{aligned}
&\Phi_1^1(r,z)=\int_0^\infty [A\,\mathrm{sh}\lambda z+B\,\mathrm{ch}\,\lambda z]J_0(\lambda r)d\lambda\\
&\Phi_1^2(r,z)=\int_0^\infty [C\,\mathrm{sh}\lambda z+D\,\mathrm{ch}\,\lambda z]J_0(\lambda r)\lambda d\lambda\\
&\Phi_2^1(r,z)=\int_0^\infty Ee^{\lambda z}J_0(\lambda r)d\lambda,\quad \Phi_2^2(r,z)=\int_0^\infty Fe^{\lambda z}J_0(\lambda r)\lambda d\lambda
\end{aligned}
\tag{1.2}
$$

(where $J_n(r)$ is the cylindrical Bessel function), on the basis of the conditions (1.1) (except for the last two equations), for the six unknown functions of the parameter λ – A, B, C, D, E and F – we obtain the following system of six linear algebraic equations ($u = \lambda h$)

$$
\begin{aligned}
&2(1-\nu_1)(C\,\mathrm{ch}\,u+D\,\mathrm{sh}\,u)-A\,\mathrm{sh}\,u-B\,\mathrm{ch}\,u-u(C\,\mathrm{sh}\,u+D\,\mathrm{ch}\,u)=-Q\\
&(1-2\nu_1)(C\,\mathrm{sh}\,u+D\,\mathrm{ch}\,u)-A\,\mathrm{ch}\,u-B\,\mathrm{sh}\,u-u(C\,\mathrm{ch}\,u+D\,\mathrm{sh}\,u)=0\\
&B=E+U,\ (3-4\nu_1)D-A=(3-4\nu_2)F-E+W\\
&G_1[2(1-\nu_1)C-B]=G_2[2(1-\nu_2)F-E]\\
&G_1[(1-2\nu_1)D-A]=G_2[(1-2\nu_2)F-E]
\end{aligned}
\tag{1.3}
$$

where, by a Hankel transformation we have

$$
\begin{aligned}
&Q=\frac{1}{2G_1\lambda}\int_0^a q(\rho)J_0(\lambda\rho)\rho d\rho\\
&U=\int_a^b u(\rho)J_1(\lambda\rho)\rho d\rho,\quad W=\int_a^b w(\rho)J_0(\lambda\rho)\rho d\rho
\end{aligned}
\tag{1.4}
$$

The determinant of system (1.3) is a known function [7] equal (apart from an unimportant factor) to

$$
\begin{aligned}
&\Delta(u)=M-(1+4u^2+LM)e^{-2u}+Le^{-4u}\\
&L=\frac{G_1\varkappa_2-G_2\varkappa_1}{G_1\varkappa_2+G_2},\quad M=\frac{G_1+G_2\varkappa_1}{G_1-G_2},\quad \varkappa_i=3-4\nu_i,\ i=1,2
\end{aligned}
\tag{1.5}
$$

In each specific case of the values of the constants M and L, it can be verified directly that the function $\Delta(u)\neq 0$ when $u\geqslant 0$. After system (1.3) has been solved, by satisfying the remaining two conditions of (1.1), we obtain a system of integral equations in the functions $u(r)$ and $w(r)$

$$
\begin{aligned}
&\int_a^b u(\rho)K_{2-l,1}^{2,l}(r,\rho)\rho d\rho+\int_a^b w(\rho)K_{2-l,0}^{2,l+1}(r,\rho)\rho d\rho=(-1)^l\frac{h(1-\nu_1)}{G_1}F_{2-l,0}^{1,l}(r)\\
&(a\leqslant r\leqslant b,\ l=1,2)
\end{aligned}
\tag{1.6}
$$

where the following notation is introduced

$$
\begin{aligned}
&K_{mn}^{kl}(r,\rho)=\int_0^\infty J_m\left(\frac{ur}{h}\right)J_n\left(\frac{u\rho}{h}\right)\frac{u^k}{\Delta(u)}K_l(u)du\\
&K_s(u)=A_0+[A_1+(2-s)A_2u+2A_1u^2]e^{-2u}-(A_0+A_1)e^{-4u},\quad s=1,3\\
&K_2(u)=B_0+(B_1+B_2u^2)e^{-2u}+B_0e^{-4u}\\
&A_0=\frac{1+M(L-2)}{2},\quad A_1=M-L,\quad A_2=2(L-1)(M-1)
\end{aligned}
\tag{1.7}
$$

$$B_0=\frac{LM-1}{2},\ B_1=1-\frac{L^2+M^2}{2},\ B_2=4-2(L+M)$$

$$F_{mn}^{kl}(r)=\int_0^c q(\rho)\rho d\rho\int_0^\infty J_m\left(\frac{ur}{h}\right)J_n\left(\frac{u\rho}{h}\right)\frac{u^k}{\Delta(u)}f_l(u)du$$

$$f_l(u)=[2(l-1)M-(-1)^l LM-1+2(M-1)u]e^{-u}+$$

$$+(-1)^l[2(l-1)L-(-1)^l LM-1-2(L-1)u]e^{-3u}$$

2. THE ASYMPTOTIC SOLUTION FOR THE CASE OF A NARROW ANNULAR CRACK

Taking into account the formulae [8]

$$\int_0^r J_0\left(\frac{ur}{h}\right)r\,dr=\frac{hr}{u}J_1\left(\frac{ur}{h}\right),\ \int_0^r J_1\left(\frac{ur}{h}\right)dr=\frac{h}{u}\left[1-J_0\left(\frac{ur}{h}\right)\right] \tag{2.1}$$

and the fact that $u(a)=u(b)=w(a)=w(b)=0$, we will integrate by parts the left-hand sides of system (1.6) with respect to the variable ρ and apply the operator $\int_0^r\ldots dr$ to Eq. (1.6) when $l=1$ and the operator $-\frac{1}{r}\int_0^r)\ldots r\,dr$ when $l=2$.

Further, we will confine ourselves to the important special case when $q(r)=q=\text{const}$ and introduce dimensionless notation according to the following formulae (E_1 and E_2 are unknown integration constants)

$$\lambda=2\left(\ln\frac{b}{a}\right)^{-1},\ \mu=\frac{h}{a},\ c_*=\frac{c}{h},\ q_*=\frac{qc\lambda}{\theta_1 a},\ E_1^*=\frac{E_1\lambda}{q_*\mu a^2},\ E_2^*=\frac{E_2\lambda}{q_*\mu a^3} \tag{2.2}$$

$$x=\lambda\ln\frac{r}{a}-1,\ \xi=\lambda\ln\frac{\rho}{a}-1,\ \varphi_1(x)=\frac{[u(r)r]'r^{1/2}}{q_*a^{1/2}},\ \varphi_2(x)=\frac{w'(r)r^{3/2}}{q_*a^{3/2}}$$

After reduction, we will obtain the following system of equations ($|x|\leqslant 1$)

$$\int_{-1}^{1}\varphi_1(\xi)K_{l-1,0}^{l}(x,\xi)d\xi-\int_{-1}^{1}\varphi_2(\xi)K_{l-1,1}^{l+1}(x,\xi)d\xi=$$

$$=(-1)^{l-1}\pi\left\{F_{l-1}^{l}(x)+\mu E_l^*\exp\left[(-1)^{l-1}\frac{1+x}{2\lambda}\right]\right\} \tag{2.3}$$

the kernels and functions on the right-hand side of which have the form

$$K_{mn}^{k}(x,\xi)=\pi\exp\frac{2+x+\xi}{2\lambda}\int_0^\infty\frac{K_k(u)}{\Delta(u)}J_m\left(\frac{u}{\mu}\exp\frac{1+x}{\lambda}\right)J_n\left(\frac{u}{\mu}\exp\frac{1+\xi}{\lambda}\right)du \tag{2.4}$$

$$F_m^l(x)=-\mu\exp\frac{1+x}{2\lambda}\int_0^\infty\frac{f_l(u)}{u\Delta(u)}J_m\left(\frac{u}{\mu}\exp\frac{1+x}{\lambda}\right)J_1(uc_*)du$$

To solve system (2.3), (2.4) we will apply the asymptotic "large λ" method [1], which is effective for a fairly narrow annular crack and a relatively thick elastic layer ($h>b$). By separating out the principal parts when $u\to\infty$ in the symbols of kernels (2.4) according to the formulae

$$\frac{K_s(u)}{\Delta(u)}=\frac{A_0}{M}+L_s(u),\ s=1,3;\ \frac{K_2(u)}{\Delta(u)}=\frac{B_0}{M}+L_2(u) \tag{2.5}$$

we can represent the functions $K_{mn}^k(x,\xi)$ of the form (2.4) as a combination of series in powers of the parameter λ^{-1}. Expansions corresponding to the first term on the right-hand side of the first formula of (2.5) are obtained by means of expansions for the functions $l(t)$ ([9, pp. 17–19]) and $l_1(t)$ ([9, p. 99]) and converge when $\lambda > 2/\pi$. The expansion corresponding to the first term on the right-hand side of the second formula of (2.5) is found using the integral [8]

$$\int_0^\infty J_0(ur)J_1(u\rho)du = \begin{cases} \rho^{-1}, & \rho > r \\ 0, & \rho < r \end{cases}$$

and converges for any λ. It can be shown that expansions in Maclaurin series containing the functions $L_k(u) \sim \exp(-2u)(u \to \infty)(k = 1, 2, 3)$ converge when $\mu > 1$ and $\lambda > 2/\ln \mu$, that is, when $h > b$.

We will seek the pair of functions $\varphi_l(x)(l = 1, 2)$ in the form

$$\varphi_l(x) = \sum_{m,n=0}^{\infty} \lambda^{-m} \ln^n \lambda \varphi_l^{mn}(x) \tag{2.6}$$

By virtue of the above estimates, series (2.6), as can be shown [9–11], are at least asymptotic within the region $\Omega = \{\mu > 1, \lambda > \max(2/\pi, 2/\ln \mu)\}$ in the μ, λ plane.

Introducing expansions of the kernels $K_{mn}^k(x, \xi)$ and representation (2.6) into Eqs (2.3), expanding the right-hand sides of Eqs (2.3) in powers of λ^{-1}, and equating terms with like factors $\lambda^{-m} \ln^n\lambda$, we obtain a chain of successively solvable systems of equations

$$\int_{-1}^{1} \varphi_l^{mn}(\xi)\left[-\ln\left|\frac{\xi - x}{\lambda}\right| + a_{00}^{(l)}\right]d\xi + (-1)^l \frac{\pi}{2}\varepsilon \int_{-1}^{1} \varphi_{3-l}^{mn}(\xi)\operatorname{sgn}\left(\frac{\xi - x}{\lambda}\right)d\xi =$$

$$= \pi f_l^{mn}(x) + \frac{\pi}{2}\varepsilon c_{00} P_{3-l}^{mn} \quad (|x| \leqslant 1,\ l = 1, 2) \tag{2.7}$$

Here

$$\varepsilon = \frac{B_0}{A_0},\quad P_l^{mn} = \int_{-1}^{1} \varphi_l^{mn}(x)dx$$

$$a_{00}^{(l)} = 2(2 - l) + 0{,}07944 + \frac{\pi M}{\mu A_0}\int_0^\infty L_{2l-1}(u)J_{l-1}^2\left(\frac{u}{\mu}\right)du$$

$$c_{00} = 1 + \frac{2M}{\mu B_0}\int_0^\infty L_2(u)J_0\left(\frac{u}{\mu}\right)J_1\left(\frac{u}{\mu}\right)du$$

and the functions $f_l^{mn}(x)(m + n \geqslant 1)$ depend on the functions $\varphi_l^{jk}(x)(l = 1, 2)$ determined from the previous systems (the constants E_1^* and E_2^* occur in them linearly).

The real integral equations (2.7) can be represented in the form of a single complex equation ($|x| \leqslant 1$)

$$\int_{-1}^{1} \varphi^{mn}(\xi)\left[-\ln|\xi - x| - i\frac{\pi}{2}\operatorname{th}(\pi A)\operatorname{sgn}(\xi - x)\right]d\xi = \pi f^{mn}(x) + Q^{mn} \tag{2.8}$$

$$Q^{mn} = -(\ln\lambda + a_{00}^{(1)})P_1^{mn} + \tfrac{1}{2}\pi\varepsilon c_{00}P_2^{mn} + i[\tfrac{1}{2}\pi\varepsilon c_{00}P_1^{mn} - (\ln\lambda + a_{00}^{(2)})P_2^{mn}]$$

$$\varphi^{mn}(x) = \varphi_1^{mn}(x) + i\varphi_2^{mn}(x),\quad f^{mn}(x) = f_1^{mn}(x) + if_2^{mn}(x),\quad A = \frac{1}{2\pi}\ln\frac{1 - \varepsilon}{1 + \varepsilon}$$

It can be proved that, if the Poisson's ratios of the elastic bodies $\upsilon_n \in (0, 1/2](n = 1, 2)$, then the constant $\varepsilon \in (-1/2, 1/2)$. Differentiating both sides of Eq. (2.8) with respect to x, we will have the integral equation ($|x| \leqslant 1$)

$$\int_{-1}^{1} \frac{\varphi^{mn}(\xi)}{\xi - x}d\xi + i\pi\operatorname{th}(\pi A)\varphi^{mn}(x) = \pi[f^{mn}(x)]' \tag{2.9}$$

the solution of which is known in closed form [1, 12]

$$\varphi^{mn}(x)=\frac{\mathrm{ch}(\pi A)}{\pi S(x)}\left[P^{mn}-\mathrm{ch}(\pi A)\int_{-1}^{1}\frac{[f^{mn}(\xi)]'S(\xi)}{\xi-x}d\xi\right]+\frac{i}{2}\mathrm{sh}(2\pi A)[f^{mn}(x)]' \tag{2.10}$$

$$S(x)=(1+x)^{1/2+iA}(1-x)^{1/2-iA},\quad P^{mn}=P_1^{mn}+iP_2^{mn}$$

To determine the complex constant P^{mn}, we multiply both sides of Eq. (2.8) by $S_1(x) = (1+x)^{1/2-iA}(1-x)^{1/2+iA}$ and integrate them with respect to x in the range from -1 to 1. Using relations (1.23) and (1.24) from [1], we obtain the linear equation

$$BP^{mn}-Q^{mn}=\mathrm{ch}(\pi A)\int_{-1}^{1}\frac{f^{mn}(x)}{S_1(x)}dx \tag{2.11}$$

$$B=-\ln 2-C-\tfrac{1}{2}\psi(\tfrac{1}{2}+iA)-\tfrac{1}{2}\psi(\tfrac{1}{2}-iA)$$

where $\psi(x)$ is the psi function and C is Euler's constant. Separating the real and imaginary parts in (2.11), we obtain that the determinant of the system of equations in P_1^{mn} and P_2^{mn} is equal to

$$D=(\ln\lambda+a_{00}^{(1)}+B)(\ln\lambda+a_{00}^{(2)}+B)-\tfrac{1}{4}(\pi\varepsilon c_{00})^2 \tag{2.12}$$

The problem of investigating the zeros of this determinant in the μ, λ plane reduces to solving the quadratic equation for in λ. Numerical analysis carried out for various pairs of elastic materials indicates that in practical situations the roots of the equation $D = 0$ lie outside the region Ω.

Note that the series of integrals, including singular integrals containing the functions $S(x)$ and $S_1(x)$ encountered in formulae (2.10) and (2.11), is calculated by means of formulae (1.6) and (1.7) [13]. In particular ($|\xi|) \leqslant 1$)

$$\int_{-1}^{1}\frac{x\,dx}{S(x)}=-i\frac{2\pi A}{\mathrm{ch}(\pi A)},\quad \int_{-1}^{1}\frac{S(x)dx}{x-\xi}=-i\pi\,\mathrm{th}(\pi A)S(\xi)-\frac{\pi(\xi+i2A)}{\mathrm{ch}(\pi A)} \tag{2.13}$$

Below we will give the first few terms of expansions (2.6). In zero the approximation

$$f_l^{00}(x)=f_l^{00}+E_l^*,\quad f_l^{00}=-\frac{M}{A_0}\int_0^{\infty}\frac{f_l(u)}{u\Delta(u)}J_{l-1}\left(\frac{u}{\mu}\right)J_1(uc_*)du$$

$$\varphi^{00}(x)=\frac{\mathrm{ch}(\pi A)}{\pi S(x)}P^{00} \tag{2.14}$$

$$P_l^{00}=\frac{\pi}{D}[(\ln\lambda+a_{00}^{(3-l)}+B)(f_l^{00}+E_l^*)+\tfrac{1}{2}\pi\varepsilon c_{00}(f_{3-l}^{00}+E_{3-l}^*)]$$

For the first approximation

$$f_l^{10}(x)=-a_{10}^{(l)}(r_lP_l^{00}+2AP_{3-l}^{00})-c_{10}^{(3-l)}(r_lP_{3-l}^{00}-2AP_l^{00})+$$

$$+r_l\frac{1+x}{2}[f_l^{10}+E_l^*+2(r_la_{10}^{(l)}P_l^{00}+c_{10}^{(l)}P_{3-l}^{00})]+\varepsilon\int_x^{r_l}\varphi_{3-l}^{00}(\xi)\frac{x-\xi}{2}d\xi$$

$$a_{10}^{(l)}=\frac{M}{\mu A_0}\int_0^{\infty}L_{2l-1}(u)J_{l-1}\left(\frac{u}{\mu}\right)R_l\left(\frac{u}{\mu}\right)du \tag{2.15}$$

$$c_{10}^{(l)}=\frac{M}{\mu A_0}\int_0^{\infty}L_2(u)J_{2-l}\left(\frac{u}{\mu}\right)R_l\left(\frac{u}{\mu}\right)du$$

$$f_l^{10}=-\frac{M}{A_0}\int_0^{\infty}\frac{2f_l(u)}{u\Delta(u)}J_1(uc_*)R_l\left(\frac{u}{\mu}\right)du$$

$$r_l=(-1)^{l-1},\quad R_l(x)=\tfrac{1}{2}J_{l-1}(x)-xJ_{2-l}(x)$$

Table 1

Case	Steel on brass			Brass on steel		
μ	2	3	4	2	3	4
a_{00}^1	1.500	1.655	1.748	1.397	1.579	1.688
$a_{00}^2 \times 10^5$	–6107	3013	5724	–1168	907	4750
c_{00}	–3.525	–1.262	–0.3324	2.375	1.841	1.531
$a_{10}^1 \times 10^5$	–6433	–5601	–4712	–7611	–6558	–5534
$a_{10}^2 \times 10^5$	5433	2127	999	7351	2995	1427
$c_{10}^1 \times 10^5$	2969	2391	1631	–1233	337	457
$c_{10}^2 \times 10^5$	–18370	–9523	–5706	–4803	–3296	–2185
$f_1^{00} \times 10^4$	8418	8885	9052	3521	3801	3901
$f_2^{00} \times 10^4$	3961	2735	2076	2470	1713	1302
$f_1^{10} \times 10^4$	5114	7363	8187	1539	2892	3385
$f_2^{10} \times 10^4$	–10860	–7902	–6101	–6686	–4928	–3818

The function $\varphi^{10}(x)$ is found by means of formulae (2.10), (2.11) and (2.15) using integrals (2.13). Integrals with a variable limit of integration in formulae (2.15) must be found numerically.

Table 1 gives values of the constants which occur in formulae (2.14) and (2.15), with $c_* = 1$ and various μ, for two cases: steel on brass and brass on steel. For steel it was assumed [14] that Young's modulus $E \times 10^{-4} = 200$ kg/cm^2 and Poisson's ratio $\upsilon = 0.28$ (shear modulus $G = E/[2(1 + \upsilon)]$) and for brass $E \times 10^{-4} = 90$ kg/cm^2 and $\upsilon = 0.35$.

The required functions $u(r)$ and $w(r)$ are expressed in terms of the functions $\varphi_1(x)$ and $\varphi_2(x)$, according to relations (2.2), by means of the formulae

$$u(r) = \frac{1}{r}\int_a^r [u(\rho)\rho]' d\rho = \frac{qca}{\theta_1 r} I_1(r), \quad w(r) = \int_a^r w'(\rho) d\rho = \frac{qc}{\theta_1} I_2(r)$$
$$I_l(r) = \int_{-1}^{\lambda \ln(r/a)-1} \varphi_l(x)\exp\left(r_l \frac{1+x}{2\lambda}\right)dx \tag{2.16}$$

Confining ourselves in series (2.6) to a finite number of terms, in accordance with the accuracy required we finally determine the constants E_1^* and E_2^* from the conditions $u(b) = w(b) = 0$ (the conditions $u(a) = w(a) = 0$ on the basis of the formulae (2.16) are already satisfied). From the formulae (2.16) we obtain a system of linear algebraic equations for E_1^* and E_2^*:

$$\int_{-1}^{1} \varphi_l(x)\exp\left(r_l \frac{1+x}{2\lambda}\right)dx = 0, \quad l = 1, 2 \tag{2.17}$$

Oscillations of the functions $\varphi_l(x)(l = 1, 2)$ in the neighbourhood of the points $x = \pm 1$ [see (2.10)] lead to oscillation of the functions $u(r)$ and $w(r)$ in the neighbourhood of the contours $r = a + 0$ and $r = b - 0$, and also to oscillation of the stresses σ_z and τ_{rz} in the neighbourhood of the contours $r = a - 0$ and $r = b + 0$.

3. THE CONDITIONS FOR DETERMINING THE DIMENSIONS OF THE ANNULAR CRACK

After the system of integral equations (1.6) has been solved, and the asymptotic forms of the stresses in the layer and half-space with respect to r in the neighbourhood of the contours $r = a - 0$ and

$r = b + 0$, have been constructed the problem of determining the internal radius a and the external radius b of the annular crack as a function of the load $q(r)$ can be solved in the following way.

We will assume that the zones of oscillation of the stresses σ_z and τ_{rz}, like the zones of oscillation of the displacements $u(r)$ and $w(r)$, are relatively small, i.e. the maximum width of the zones of oscillation $\delta_0 \ll b - a$. This, for example, should occur when the substrate (half-space) is considerably stiffer than the coating (layer) or vice versa (see the estimate of the width of the zones of oscillation in Section 114a in [15]).

Note that between the layer (coating) and half-space (substrate) there is always a transition zone [16] – a transition layer of thickness $\delta \ll h$ (usually several micrometres), more brittle than the main materials, with the elastic characteristics G_3 and υ_3 [$E_3 = 2G_3(1 + \upsilon_3)$] which can be obtained by a certain averaging of G_1, G_2 and υ_1, υ_2, Likewise, let $\delta > \delta_0$.

Since the crack was formed on the boundary $z = 0$ due to the inadequate strength of the given transition layer, it is highly probable that, when the load $q(r)$ increases, the crack will then spread into the plane of minimum resistance $z = 0$. By virtue of this, to determine the values of a and b it is possible, for example, to use the following additional conditions when $z = 0$

$$\frac{1}{\delta}\int_{a-\delta}^{a} U dr = A_*, \quad \frac{1}{\delta}\int_{b}^{b+\delta} U dr = A_* \tag{3.1}$$

which are certain analogues of Novozhilov's integral criterion of fracture [2]. Here, U is the elastic energy per unit volume of the transition layer, namely

$$\begin{gathered} U = \frac{1}{2E_3}[\sigma_r^{*2} + \sigma_\varphi^{*2} + \sigma_z^2 - 2\nu_3(\sigma_r^*\sigma_\varphi^* + \sigma_r^*\sigma_z + \sigma_\varphi^*\sigma_z)] + \frac{1}{2G_3}\tau_{rz}^2 \\ \sigma_r^* = \tfrac{1}{2}(\sigma_r^{(1)} + \sigma_r^{(2)}), \quad \sigma_\varphi^* = \tfrac{1}{2}(\sigma_\varphi^{(1)} + \sigma_\varphi^{(2)}) \\ \sigma_z = \sigma_z^{(1)} = \sigma_z^{(2)}, \quad \tau_{rz} = \tau_{rz}^{(1)} + \tau_{rz}^{(2)} \end{gathered} \tag{3.2}$$

and A^* is the experimentally determined physical constant of the pair of materials (the critical value of elastic energy per unit volume of the transition layer). The specific use of the criterion (3.1) is still difficult in view of the lack of the necessary experimental data.

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